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THE PREDICTION AND DISCOVERY OF CONICAL REFRACTION BY WILLIAM ROWAN HAMILTON AND HUMPHREY LLOYD (1832–1833)

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ABSTRACT

The discovery of conical refraction in biaxial crystals by Humphrey Lloyd followed a brilliant mathematical prediction by William Rowan Hamilton, which was first announced at the Royal Irish Academy in October 1832. The importance of the discovery was entirely theoretical; it represented the completion of the theory of double refraction published by Huygens in 1690. Fresnel, who developed the received theory of double refraction in biaxial crystals, had inadequately described the form and properties of the wave surface which bears his name. Hamilton discovered four conoidal cusps on the wave surface, from which he predicted the phenomena of external and internal conical refraction. The discovery was a triumph for the view and methodology of optics presented by Hamilton in a series of memoirs on the 'Theory of systems of rays'. Hamilton's correspondence at the end of 1832 and in early 1833 reveals that the discovery was made possible by close collaboration with the experimentalist Lloyd.

Theory of double refraction: Huygens and Fresnel

The phenomenon of double refraction in crystallised minerals was discovered by the Dane, Erasmus Bartholin, in about the year 1669. Bartholin found that a beam of light on being refracted at the surface of a crystal of Iceland spar (carbonate of calcium) travels through it as two pencils, one of which is refracted according to the ordinary law (Snell's law) and the other according to a new or extraordinary law. A few years later Christiaan Huygens discovered the different polarisations of the two pencils and explained the phenomena on the principles of the wave theory. In the fifth chapter of his *Traité de la lumière*, published at the Hague in 1690, he described the form of the two waves in the crystal; the form of the ordinary wave was shown to be a sphere as in isotropic substances, while the extraordinary wave was shown to be a spheroid or ellipsoid of revolution. The sphere of ordinary refraction lies within the spheroid touching it at two points, the extremities of the 'optic axis' which represents the one direction in which no double refraction occurs [12].

Although Huygens' *Traité* was only published in 1690 we know that he arrived at the explanation of double refraction in Iceland spar thirteen years earlier. The

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manuscript containing his drawings and calculations is preserved at the Museum Boerhaave, in Leiden. At the top of the page Huygens wrote

EYPHKA 6 Aug. 1677,

echoing Archimedes' alleged cry on discovering his principle [2, p.213].

Huygens' construction for obtaining the refracted rays in Iceland spar was thoroughly investigated in the nineteenth century, notably by George Gabriel Stokes, Richard Glazebrook and C. S. Hastings, all of whom confirmed the accuracy of the construction to within the limits of experimental error [22], [6], [11]. Huygens' law was found to apply to other uniaxial crystals such as quartz but in this case the spheroid lies inside the sphere. Moreover, David Brewster discovered in 1813 that the mineral topaz has two axes of no double refraction and subsequently others such as arragonite,* borax and mica were identified as biaxial. The discovery of biaxial crystals meant that Huygens' law had lost its generality and a new and general theory of double refraction was wanting.

Although Huygens had provided a mathematical construction that was accurate and experimentally verified, it was not accepted by the scientific community in either England or France during the eighteenth or early nineteenth century. Huygens' explanation was intimately connected with the wave theory and was therefore unacceptable to the corpuscular-minded scientists of the post-newtonian era. In France especially corpuscular optics had become a central tenet of the physics of short-range forces developed by Pierre Simon Laplace and his school [3]. Furthermore, Huygens had been unable to explain the effect produced when light is passed through two crystals in succession; thus when the ordinary and extraordinary rays are made to pass through a second crystal of Iceland spar they are in general both divided into two further rays. However, for a certain orientation of the crystals the rays do not undergo any further division, although when a ray of light is allowed to pass directly to the second crystal in this position double refraction undoubtedly occurs.

Newton, in the Queries of the third book of *Opticks*, gave a detailed description of the phenomenon of double refraction, and the inability of the Huygenian theory to explain the cause of the different polarisations of the refracted rays in Iceland spar was one of the reasons he rejected the theory. He writes:

> ... the unusual refraction of Island-Crystal appears to be due to some attractive virtue lodged in certain sides of the rays and of the particles of the crystal — this virtue seems not magnetical, but is similar — it is difficult to conceive how rays of light can have a permanent virtue in two of their sides and not in the others unless they are bodies [17, Query 29].

In 1802 William Hyde Wollaston, at the instigation of Thomas Young, re-examined and experimentally verified Huygens' construction for the extraordinary wave. This created a dilemma for the supporters of the corpuscular theory and before the end of the decade further corpuscular explanations of double refraction had been proposed by Laplace and by Etienne Louis Malus, who both derived Huygens' construction from a particle model of light. Double refraction became the subject of a prize competition of

*Arragonite is a rhombic crystal of calcium carbonate ($CaCO_3$). The word is usually spelt *aragonite* today, from the district of Aragon in north-east Spain.

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the mathematical section of the Institut de France in 1808. Malus entered the competition and in the course of preparing for it discovered the phenomenon of polarisation by reflection. He won the prize in 1810 with a two-hundred-page essay on double refraction which accorded with the orthodox Laplacian corpuscular view. This interesting episode in the history of physics has been discussed by Whittaker [23], by Ronchi [19] and more recently by Frankel [4]. However, within a decade and a half the discovery of biaxial crystals and the development of Fresnel's theory of double refraction had rendered these corpuscular explanations obsolete.

Fresnel's writings on the double refraction dating from the years 1821–1822 are published in the second volume of his collected works published in 1868 [5]. Having set out from the hypothesis that the elasticity of the vibrating medium within the crystal is unequal in three rectangular directions, he showed that the surface of the wave is neither a sphere nor a spheroid but a surface of the fourth order consisting of two sheets whose points of contact with the tangent planes determine the direction of the two refracted rays in the biaxial crystal. In general neither of the rays obeys Snell's law or that of Huygens and both are refracted according to a new and more complicated law.

Taking the elasticities in the directions of the co-ordinate axes x, y and z as a^2, b^2, and c^2 respectively, Fresnel found the equation of the wave surface to be

$$(x^2 + y^2 + z^2)(a^2x^2 + b^2y^2 + c^2z^2) - a^2(b^2 + c^2)x^2 - b^2(a^2 + c^2)y^2 - c^2(a^2 + b^2)z^2 + a^2b^2c^2 = 0.$$

When the elasticity of the medium is equal in two of the three directions the equation of the wave surface can be resolved into two quadratic factors, which give the equations of the sphere and spheroid of the Huygensian theory, the two optic axes coinciding in one in this case. Thus Huygens' law was found to be a particular case of the general law. Similarly Snell's law was deduced by taking the elasticity in all three directions as equal.

In deriving the equation of the wave surface Fresnel applied co-ordinate or Carthesian geometry. The procedure he found for obtaining this equation was long and unwieldy and in fact the full calculation was not given. It was as if the procedure was so inelegant as to be almost an embarrassment and that he was content merely to outline the method.

After Fresnel several authors published alternative methods for obtaining the wave equation and demonstrating its properties. The first was André Marie Ampère who published a long and rather complicated method of finding the equation in 1828 [1]. James MacCullagh of Trinity College Dublin in papers presented to the Royal Irish Academy in 1830 and 1833 developed a body of geometrical theorems which he then applied to the wave theory and specifically to the Fresnel wave surface which he called the 'biaxial surface' [15] [16]. The Fresnel wave surface was a topic which continued to excite the interest of mathematical physicists until the turn of the present century and beyond. During the nineteenth century about 200 works on the topic of the Fresnel wave surface were published by some one hundred authors, who included Augustin Louis Cauchy, Arthur Cayley, Franz E. Neumann, John William Strutt (Lord Rayleigh), James Joseph Sylvester and William Rowan Hamilton whose investigations led to the discovery of new properties of the wave surface which all previous investigators, including Fresnel himself, had misapprehended.

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The representation of the wave surface was also a favourite motif for textbook writers in the late nineteenth and early twentieth centuries. The *Theory of light* of Thomas Preston of Dublin gives a particularly good exposition of the properties of the wave surface. First published in 1890 it went through several editions (the fifth, 1928) and was well known throughout the English-speaking world.

The Fresnel wave surface is of the fourth order and consists of two sheets which penetrate each other. Its section with the *xz* plane shows a circle overlapping an ellipse (see Fig. 1). It has four singularities at the points where the two sheets of the wave meet (p_1, p_2, p_3, and p_4 in Fig. 1). Hamilton called the lines drawn from the origin of co-ordinates, the source from which the wave is being propagated, to each of these four points lines of 'single ray-velocity' (op_1, op_2, op_3, op_4 in Fig. 1). At each of the four singular points on the wave surface, viz. at the ends of the lines of single ray-velocity, one can draw tangent planes which give the directions of the refracted rays. Fresnel had supposed that only two such tangent planes, one to each sheet of the wave surface, could be drawn at each of the four points. Hamilton's discovery was that the four singular points are in fact trumpet-like cusps. The lines of single ray-velocity were then called 'cusp rays' and at the end of each the wave surface is touched not by only two tangent planes but by a tangent cone.

According to Fresnel's representation each of the four singularities can be covered over and enclosed by tangent planes. In his conception each of the tangent planes touches the wave surface at two points only. Hamilton found, however, that these

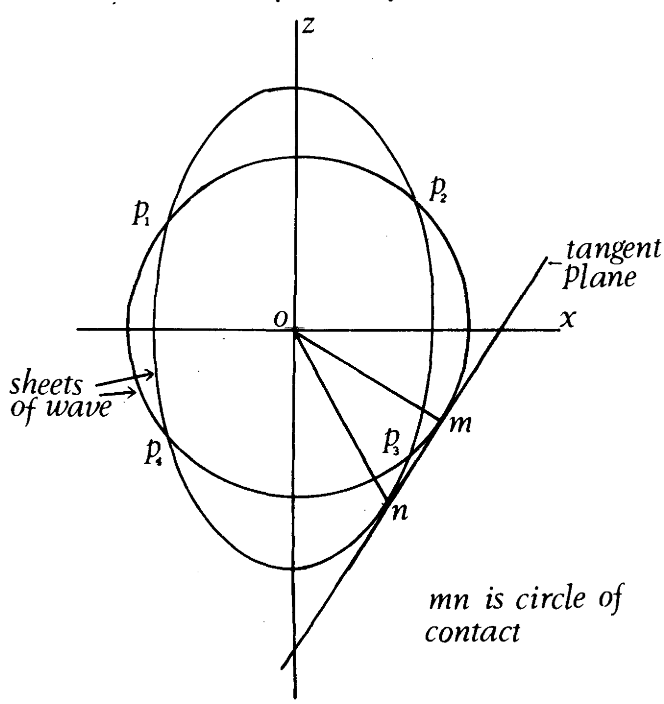

FIG. 1—Section of Fresnel wave surface with *xy* plane showing cusps (p_1, p_2, p_3, p_4) and tangent of circular contact (*mn*).

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tangent planes touch the wave surface along circles of contact. In Fig. 1 *mn* is such a circle of contact. The perpendiculars from the origin on these four planes represent the lines of 'single normal-velocity' in Hamilton's theory. From this discovery Hamilton predicted two hitherto unknown optical phenomena in which light should be reflected as a cone on entering and leaving a biaxial crystal. In the first case, a ray passing through the crystal in a direction corresponding to that of a cusp ray will be refracted as a cone on emerging at the plane interface. In the second, a ray incident in a direction such that the refracted wave front is parallel to *mn* in Fig. 1, viz. that the wave is propagated along a line of single normal velocity, will be refracted as a cone within the crystal.

Figs 2 and 3 show the experimental arrangements by which the two phenomena were observed. In Fig. 2 the two parallel plane faces of the crystal are covered by metal plates. The adjustment is completed when the line joining the two apertures in the plates

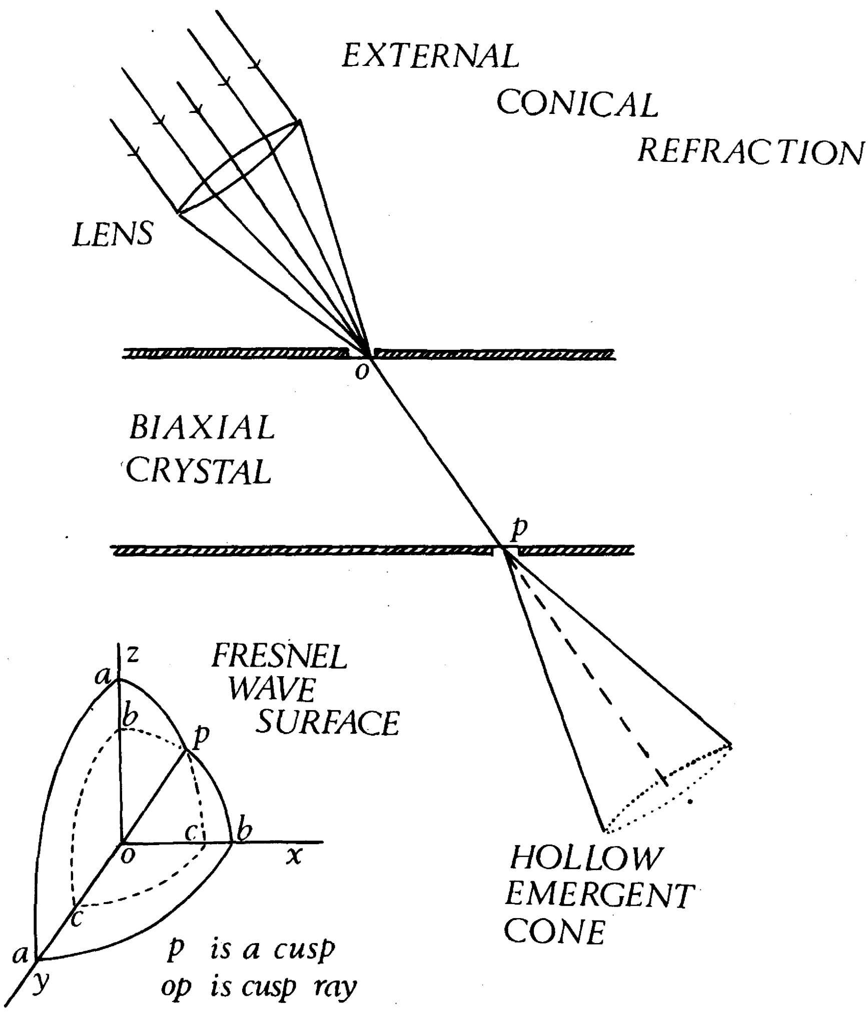

FIG. 2—Experimental arrangement to observe external conical refraction. 3-D representation of wave surface showing cusp ray.

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corresponds with the direction *op*, that of the cusp ray. A beam of light is made to converge to the point *o* using a lens; it is refracted as a single ray in the crystal and emerges at *p* as a hollow cone. In Fig. 3, the experimental arrangement by which internal conical refraction was observed is shown. The incident light from a lamp placed at a distance is made to pass through two small apertures; the first was in a screen placed close to the flame and the second in a plate of thin metal in contact with the face of the crystal. Before the adjustment is completed the incident ray is refracted as two rays in the crystal which emerge parallel at the second surface. The angle of incidence is then varied until the two rays become a continuous circle of light which emerges as a hollow cylinder at the second suface.

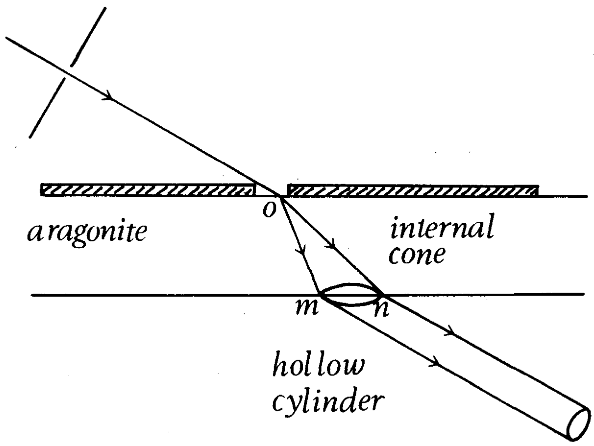

FIG. 3—Experimental arrangement to observe internal conical refraction. *mn* corresponds to the direction of the cusp ray.

The prediction of conical refraction by William Rowan Hamilton

A comprehensive account of Hamilton's achievements in the field of optics, together with bibliographical information, has been given by Hankins in recent publications [9], [10]. Between 1827 and 1833 Hamilton published a series of papers in the *Transactions of the Royal Irish Academy*. In this 'Essay on the theory of systems of rays' and in three supplements he developed his general view of optics. In particular, in the 'Third supplement', presented on 23 January and on 22 October 1832, he set out a system of general methods for the solution of optical problems, together with some general results deduced from the fundamental formula and view of optics set out in the main essay [8]. On the latter date he made the first announcement of his theoretical discovery of two new optical phenomena, viz. internal and external conical refraction in biaxial crystals. This 'Third supplement', which like most of Hamilton's writings is characterized by great generality and a high degree of mathematical abstraction, is 144 quarto pages long and is divided under thirty-one headings. In the introduction he invites attention particularly to the last five headings:

> Of these the theory of external and internal conical refraction, deduced by my general methods from the principles of Fresnel, will probably be thought the least undeserving of attention.

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Before examining the specific ideas presented in these five sections it must be remarked that Hamilton's mathematical investigations in optics are entirely analytical and all recourse to geometry is avoided. A further point to note is that reference is made to the printed version of the 'Third supplement' which appeared in the volume of the *Transactions* completed in 1837. As will be seen from the correspondence presented in the next section, the paper had undergone considerable revision since its presentation in 1832.

In the analytical table of contents Hamilton enunciated his discovery (headings 27, 28, and 29) in these terms:

27. *Theory of Fresnel.* New formulae, founded on that theory, for the velocities and polarisations of a plane wave or wave-element. New method of deducing the equation of Fresnel's curved wave, propagated from a point in a uniform medium with three unequal elasticities. Lines of single ray-velocity, and of single normal-velocity, discovered by Fresnel ...

28. *New properties of Fresnel's wave.* This wave has four conoidal cusps, at the ends of the lines of single ray-velocity; it has also four circles of contact, of which each is contained on a touching plane of single normal-velocity. The lines of single ray-velocity may therefore be called cusp-rays; and the lines of single normal-velocity may be called normals of circular contact ...

29. *New consequences of Fresnel's principles.* It follows from those principles that crystals of sufficient biaxial energy ought to exhibit two kinds of conical refraction, an external and an internal; a cusp-ray giving an external cone of rays, and a normal of circular contact being connected with an internal cone ...

Hamilton set out by deriving the equation of the Fresnel wave surface with respect to the coincident axes of co-ordinates (x, y, z) and of elasticity (a, b, c), expressing it precisely in the form given above. However, he believed his method 'will perhaps be thought simpler than that which was employed by the illustrious discoverer, and that of others which have since been proposed'.

He then wrote the wave equation in two equivalent polar forms, with respect to the same origin, in which a radius vector is expressed as a function of the angles made with two constant radii. These constant radii are the lines of single ray-velocity and single normal-velocity, respectively, of Fresnel's theory. Then, in order to investigate the properties of the wave near the end of these lines of single ray-velocity and single normal-velocity, the origin is transferred to each of these points in turn using formulae of transformation. The polar equation of the Fresnel wave surface is then written in equivalent forms which reveal the exact nature of the wave surface at these points. He thus deduced that at the ends of the lines of single ray-velocity there are conoidal cusps (four in number) and that the wave at each of these points is touched not by one tangent plane but by a tangent cone. Thus Hamilton arrived at the following conclusion:

> Fresnel does not appear to have been aware of the existence of this tangent cone to his wave; he seems to have thought that at the end of a radius ρ' of single ray-velocity, the wave was touched only by two right lines, contained in the plane of *ac*, namely, by the tangents to a certain circle and ellipse, the intersections of the wave with that plane: but it is evident from the foregoing transformation that every other section of the wave, made by a plane containing the radius vector ρ', is touched, at the end of that radius, by two tangent lines contained on the cone. It is evident also

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that there are *four such conoidal* cusps, at the ends of the four lines of single ray-velocity, $\pm \rho'$, $\pm \rho''$.

They are determined by the following co-ordinates, referred to the axes of elasticity:

$$x = \pm c \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}, y = 0, z = \pm a \sqrt{\frac{b^2 - c^2}{a^2 - c^2}};$$

and there are four intersections of fresnel's circle and ellipse, in the plane of ac, which have for their equations in that plane

$$x^2 + z^2 = b^2, \ a^2x^2 + c^2z^2 = a^2c^2.$$

The second conclusion was stated as follows:

It is evident that there are *four such circles of plane contact at the ends of the four lines* $\pm \omega'$, $\pm \omega''$, *of single normal velocity*. They are all equal to each other, and the common magnitude of their diameters is $b^{-1} \sqrt{a^2 - b^2}. \sqrt{b^2 - c^2}$. The same conclusion may be drawn from Fresnel's equation of the wave in co-ordinates xyz referred to the axes of elasticity: the equations of the *four planes of circular contact* being, in these co-ordinates,

$$z \sqrt{b^2 - c^2} \pm x \sqrt{a^2 - b^2} = \pm b \sqrt{a^2 - c^2}.$$

Fresnel, however, does not appear himself to have suspected the existence of these circles of contact, nor do they since seem to have been perceived by any other person.

In the 'Theory of systems of rays' and its supplements Hamilton introduced his well known 'characteristic function' (V), representing the optical length of a ray as a function of variable initial and final points. Now in investigating the properties of the Fresnel wave surface he introduced so-called 'components of normal slowness' of wave propagation. These components (σ, τ, υ) are the partial differential coefficients, of first order, of the characteristic function with respect to the co-ordinates x, y, z. Similarly, in the course of the 'Third Supplement' Hamilton had derived general formulae of reflection and refraction from the characteristic function which were now combined with the principles of Fresnel. He formulated a law for the refraction at the surface of a biaxial crystal. He found that in general, for both internal and external refraction at the surface of the crystal, two rays are produced from one on crossing the interface. There are, however, two exceptions to this general law of double refraction which Hamilton stated as follows (the direction of the internal ray is given by the cosines α, β, γ, and that of the external ray by α_0, β_0, γ_0):

But there are two remarkable exceptions, connected with the two sets of lines of single velocity, and with the conoidal cusps and circles of contact of Fresnel's wave. For we have seen that at a conoidal cusp the tangent plane to the wave is indeterminate; it is evident therefore that a *cusp-ray* must correspond to an infinite variety of systems of direction cosines $\alpha_0, \beta_0, \gamma_0$ of the external ray; so that *this one internal cusp-ray must correspond to an external cone of rays, according to a new theoretical law of light*, which may be called EXTERNAL CONICAL REFRACTION. And again, at a circle of contact, the wave has one common tangent plane for all the points of the circle, and therefore the infinite variety of internal rays which correspond to these different points have all one common wave normal, which may be called a *normal of circular contact*, and all these internal rays have one common system of components of normal slowness σ, τ, υ within the crystal, and

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consequently correspond to one common external ray; so that *this one external ray is connected with an internal cone of rays, according to another new theoretical law of light* which may be called INTERNAL CONICAL REFRACTION.

Having announced his theoretical discoveries at the Royal Irish Academy on 22 October 1832, Hamilton asked his colleague Humphrey Lloyd to undertake the necessary experiments in order to observe the phenomena. Lloyd set about this task at once and after some initial setbacks succeeded in observing external conical refraction on 14 December with a specimen of arragonite obtained from the firm of Dollond, London. Internal conical refraction was observed early in the new year and in February 1833 the first account of the discovery was published in the *Philosophical Magazine* [13].

Lloyd described two experimental arrangements by which he had been able to observe external conical refraction. He explained his observations as follows:

> The phaenomenon which presented itself, on looking through the aperture, when the adjustment was complete, was in the highest degree curious. There appeared a luminous circle with a small dark space round the centre, and in this dark space (which was also nearly circular) were two bright points divided by a narrow and well-defined dark line. When the aperture in the plate was slightly shifted, the appearances rapidly changed. In the first stage of its change the central dark space became greatly enlarged, and a double cone appeared within it. The circle was reduced to about a quadrant, and was separated by a dark interval from the cone just mentioned. The remote cone then disappeared, and the circular arch diminished; and as the obliquity of the line to the axis was further increased, these two luminous portions merged gradually into the two pencils into which a single ray is divided in the other parts of the crystal.

Lloyd also succeeded in projecting the external cone on a glass screen using direct sunlight. Examining the emergent cone with a tourmaline plate, he discovered that all the rays of the cone were polarised in different planes. He writes:

> I was surprised to observe that one radius only of the section of the cone vanished, in a given position of the axis of the tourmaline; and that the ray which disappeared ranged through 360°, as the tourmaline plate was turned through 180° ..... On examining the curious phenomenon more attentively, I discovered the remarkable law, — that *the angle between the planes of polarisation of any two of the rays of the cone is half the angle contained by the planes passing through the rays themselves and its axis.*

Under heading 30 of the 'Third supplement' Hamilton developed the law of conical polarisation discovered by Lloyd from his theory of conical polarisation. The final section of the 'Third supplement' is enunciated as follows:

> 31. In any uniform medium, the curved wave propagated from a point is connected with a certain other surface, which may be called the surface of components, by relations discovered by M. Cauchy, and by some new relations connected with a general theorem of reciprocity.
> This new theorem of reciprocity gives a new construction for the wave in any undulatory theory of light; and it connects the conoidal cusps and circles of contact of Fresnel's wave, with circles and cusps of the same kind upon the surface of components .....

The theorem of reciprocity referred to by Hamilton bears a close resemblance to a theory of reciprocal surfaces in the papers 'On the double refraction of light in a

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crystallized medium according to the principles of Fresnel' and 'Geometrical propositions applied to the wave theory of light', by MacCullagh [15] [16]. In the 'Third supplement' Hamilton explains that it was in contemplating this general theorem that he discovered the circles of contact on the wave surface:

> It follows from this general theory of reciprocal surfaces, that a conoidal cusp on any surface *A* corresponds in general to a curve of plane contact on the reciprocal surface, *B*, and reciprocally; and, accordingly the cusps and circles on Fresnel's wave are connected with circles and cusps on the corresponding surface of components, which latter surface is indeed deducible from the former by merely changing the semiaxes of elasticity *a b c* to their reciprocals. And it was in fact by this general theorem that I was led to discover the four circles of contact on Fresnel's wave, by concluding that the wave must touch four planes in curves instead of points of contact, as soon as I had perceived the existence of four conoidal cusps on the surface of components, by obtaining ... the formula ..., which is the approximate equation of such a cusp. I easily found also that there were *only four* such cusps on each of the two reciprocal surfaces, and therefore concluded that there were *only four*

Hamilton attributed the discovery of the surface of components to Cauchy, although he claimed to have encountered it independently through his own investigations. The theorem of reciprocity leads to a new construction for the wave surface. When a ray passes from air to a crystal, one constructs the surfaces of wave slowness for the two media with the point of incidence as the common centre. The incident ray is then produced to meet the sphere, which represents the normal slowness of the wave in air; from the point of intersection a perpendicular is drawn to the reflecting or refracting suface. This will cut the surface of slowness of the reflected or refracted waves in general in two points. The lines connecting these points with the centre represent the direction and normal slowness of the waves. On the other hand the perpendiculars from the centre on the tangent planes at these same points represent direction and slowness of the rays.

This construction, known as Hamilton's construction, gained currency abroad after Humphrey Lloyd had given an account of it in his 'Report on the progress and present state of physical optics' presented to the British Association for the Advancement of Science in 1834 and published in the Association's report for that year. This construction can be regarded as a generalization of Huygens' construction for biaxial crystals, or in the words of Lloyd, 'a very elegant construction for the reflected or refracted ray, which is, in most cases, more convenient than that of Huygens.'

Hamilton's correspondence in connection with the discovery of conical refraction in 1832–1833

In the weeks and months following the announcement of the theoretical discovery of conical refraction at the Royal Irish Academy on 22 October 1832, Hamilton exchanged a series of letters with George Biddell Airy and John F. W. Herschel in England, and especially with his colleague Humphrey Lloyd. These letters along with a great quantity of other scientific and general correspondence were assembled and arranged by his first biographer, Robert Percival Graves, and have been preserved by either the library (manuscripts department) of Trinity College Dublin or by the O'Regan

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family, who are connected with the Hamiltons in consequence of the marriage of his daughter. The letters in possession of the O'Regan family have been copied with the permission of the present owner by the library of Trinity College.

A selection from the correspondence was published by Graves in his biography of Hamilton one hundred years ago [7]. Graves was not a mathematician and presented only some non-mathematical extracts from the correspondence. In his own words:

> ... being full of mathematical formulae, they (the letters in question) are most suited for a collection of the scientific correspondence of the subject of this memoire, which I hope may some day see the light, than for the present work. Here it must suffice to give an outline of their contents, indicating the history of the discovery and its verification, and one or two letters of general statement.

Later writers on the topic of the discovery of conical refraction have relied on Graves. George Sarton marked the hundredth anniversary of the discovery with an article written on the basis of the published papers and the letters given in the Graves biography. This article was accompanied by a facsimile reproduction of Lloyd's first paper in the *Philosophical Magazine* [21].

In order to reach a full appreciation of the respective contributions of Hamilton and Lloyd in making this discovery it was necessary to re-examine and to assess all available correspondence [18, chapter 3]. The most pertinent elements of this correspondence are reproduced here. Hamilton's 'Third supplement', like much of his other published work, is characterized by a high degree of mathematical abstraction so as to be almost incomprehensible to all except mathematicians. It is a reasonable conjecture that the 'Third supplement' was never comprehended in all its details and all its subtleties by contemporaries like Lloyd, Airy and Herschel. The letters may show at least that the most important results were understood and applied with great success.

The first letter of the series, written by Hamilton to Lloyd from the observatory at Dunsink on 3 November 1832 (Hamilton-O'Regan Manuscript No. 293), appears to be a reply to a query from the latter concerning the angle of the cone in external conical refraction.

> Your three indices 1,5326; 1,6863; 1,6908; I take to be in Fresnel's theory
>
> $$\frac{1}{a}, \frac{1}{b}, \frac{1}{c};$$
>
> especially as these numbers give
>
> $$\sqrt{\frac{c^{-2}-b^{-2}}{b^{-2}-a^{-2}}} = \tan 9^\circ\, 56'\, 27'',$$
>
> and therefore the angle between the optic axes = 19° 52′ 54″, which agrees with your angle 19°53′. The angle is bisected by a in the present case of arragonite, whereas in topaz the acute angle of the optic axes is bisected by C.

The first angle here is the angle of internal incidence of the cusp ray and is half the angle between the optic axes. This letter is accompanied by the two figures presented here as Figs 4 and 5.

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The letter continues:

Let the axis α bisecting 19°53′ be the normal to the plane face of the arragonite at which the ray *i* emerges into air, having proceeded within the crystal in the direction of one of the two optic axes not from an external point but from a luminous point *L*, inside or on the surface. I think the luminous point might be the image of the sun formed on the surface by a lens of short focus. Thus I conclude from Fresnel's theory that in the plane *aC*, that is in the plane of the two optic axes represented by the plane of the paper, there ought to be two emergent rays, one of which we may call the *ordinary ray*, represented by the ordinary law of sines with the mean index 1,6863 so that its angle of internal incidence being 9°56′27″ its angle of emergence is 16°55′27″, and the other *e*, which we may call the *extraordinary ray*, having its angle of emergence = 13°54′49″, which is less than the other by 3°0′38″; and that besides there ought to be an infinite number of emergent rays out of the plane of the paper, and composing with the two already mentioned a nearly circular cone: the greatest angular deviation from the plane of the paper, that is, from the plane of the optic axes, being 1°28′24″.

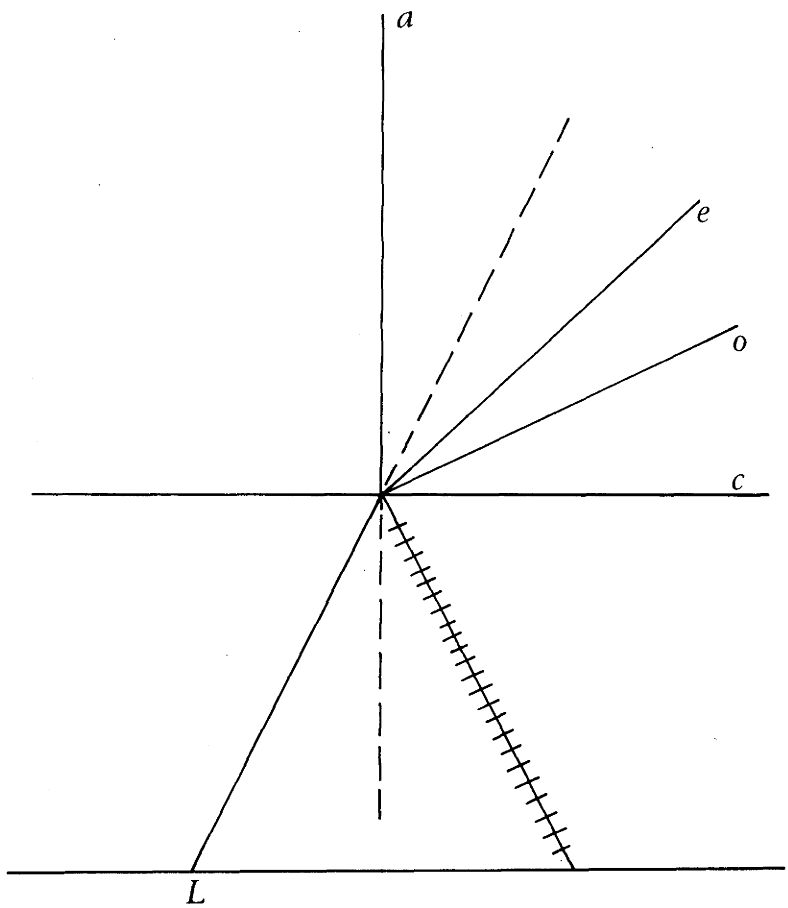

FIG. 4—Diagram copied from Hamilton-O'Regan Manuscript No. 293.

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Now take the more general case in which the normal *n* is inclined to *a* in the plane of the optic axes so that the internal angle of incidence is not 9°56′27″ but *I*; there will still be an emergent cone corresponding to some internal ray *i*, and the extreme angular deviation from the plane of the optic axes will still be 1°28′24″; but the angle between the *o*, *e*, in this plane is not exactly the same as before: and we have the following formulae to determine the two angles of emergence, measured from the normal, which we may call R_o and R_e:

$$\text{Sin } R_o = 1{,}6863 \text{ Sin } I,$$
$$\text{Sin } R_e = 1{,}68708 \text{ Sin } (I - 1°44'48'').$$

For example if $I = 9°56'27''$, then $R_o = 16°55'27''$ and $R_e = 13°54'49''$ as before; if $I = 0$, so that the face is perpendicular to the optic axis *i*, then $R_o = 0$, and $R_e = 2°56'51''$ = the angle of the cone. *I* becomes negative when *n* deviates to the other side of *i*. The cone is not exactly circular as I said before: and it varies in passing from red to violet.

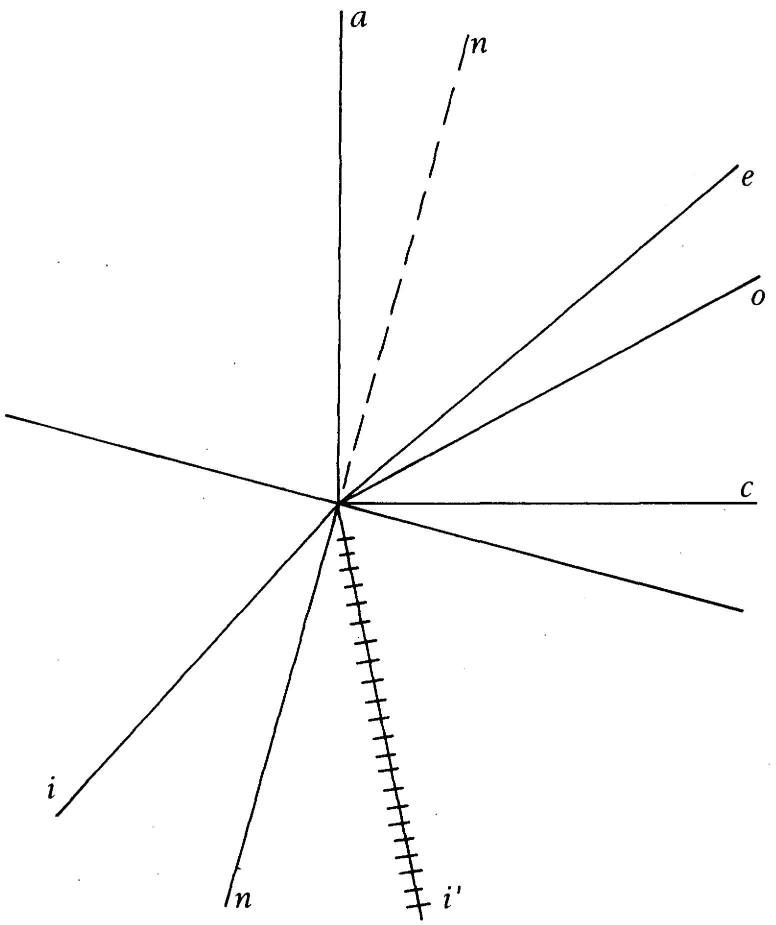

FIG. 5—Diagram copied from Hamilton-O'Regan Manuscript No. 293.

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Two days later, on 5 November, Hamilton wrote again to Lloyd on the same subject. This letter (MS No. 296) contains the drawing shown in Fig. 6. This represents the section of the face of emergence made by the plane of the optic axes which also contains the normal n. Light coming from a luminous point L within the crystal or on the surface of entry falls on this plane internally such that the emergent rays are all in the plane of the optic axes (represented by the plane of the paper) except those belonging to the two emergent cones for the two interior rays *i*, *i′*. He explains:

> Thus if a narrow slit only were left open by a piece of card on the face of emergence in the plane of the optic axes, for red light (which is not always the same as for violet) and a bright red line parallel to the slit were on or in contact with the other face of the plate in such a manner that the plane of these two parallels was perpendicular to each surface of the crystal; the red line should not be anywhere visible thro' the slit except to an eye in the plane of the line and slit, according to Fresnel's own results from his own theory; but according to my results from the same theory of Fresnel there should be two positions near which the eye tho' a little out of the plane of line and slit should see a given point of the one thro' the other; and there would in general be an infinite number of positions of the eye a little out of the same plane from each of which positions two points of the red line would be seen.
>
> This would perhaps be the easiest way of all of verifying my theoretical conclusion respecting the conical refraction. I hope you received my letter about the arragonite.

In the meantime Lloyd had undertaken the first experiments but without success; on 6 November he wrote to Hamilton the following letter (MS No. 297) which requires no further explanation:

> I have received both your communications on the subject of the arragonite. Having tried many methods of producing a delicate line of light I fixed ultimately upon the following. I covered a lens of about an inch and a quarter focus with a thin plate of copper, in the centre of which I had previously formed three small holes in the same

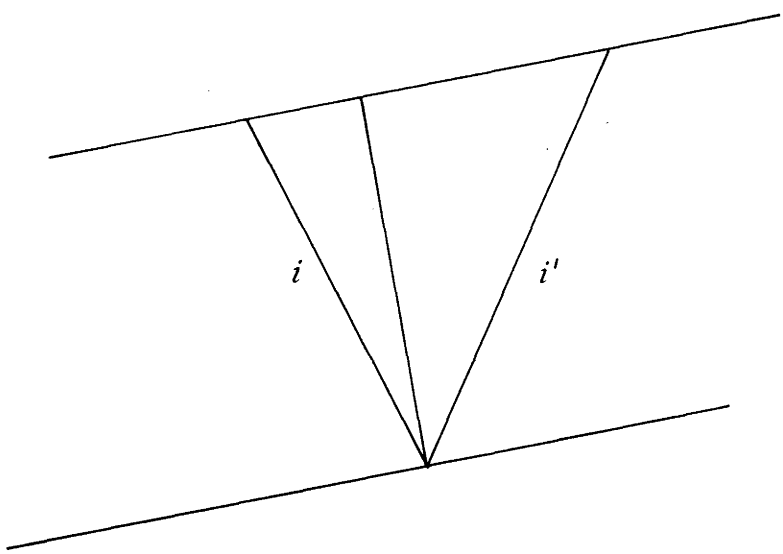

FIG. 6—Diagram copied from Hamilton-O'Regan Manuscript No. 296.

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right line, distant from each other about 1/32 nd inch, the extreme holes being distant therefore 1/16 th inch would subtend at the focus very nearly an angle of 3°.

I then took the specimen of arragonite, which is a plate cut very nearly perpendicular to the line bisecting the angle of the optic axes and placed it so that the rays proceeding from the light of a distant lamp, refracted by the lens and passing through the three holes, should converge to a point on the anterior surface as nearly as I could, having previously adjusted the inclination of the specimen (by observation of the rings) so that the ray should pass through one of the optic axes. The posterior surface of the crystal was then covered with a thin plate of copper with a minute hole in the centre. On looking through this hole at the 3 small apertures in the lens, which were in the plane passing through the optic axes of the crystal, I saw the three rays, and on repeating the experiment with the line of the three holes perpendicular to its former position, I got the same result, from which I concluded I had observed the different sections of the cone.

I was deceived however, for on endeavouring to verify the result negatively by observing in another position of the crystal not the optic axes, I found the same result. I received the emergent rays on a small screen of roughened glass as well as observing them by the naked eye. The truth is, I believe, that though I took every pains to render the holes as minute as possible, yet they bore too considerable a ratio to the thickness of the plate, which was scarcely more than 1/20 th inch, so that ordinary and extraordinary ray (of the same point) get out together without sensible separation. I must repeat the experiment in another way, and I have prepared too to try the matter in the method proposed in your note of yesterday. . . On Thursday I will let you know whether I have any success in observing the cone, but I almost despair of doing anything with so thin a plate.

On 10 November Hamilton and Lloyd exchanged letters, this time occasioned by the interest being shown by Airy in Hamilton's theoretical prediction. On 25 October Hamilton had written to Airy offering to propose him as an honorary member of the Royal Irish Academy, and had stated in that letter that he had arrived at some new results from Fresnel's theory, without revealing what these results were. On 4 November, Airy replied accepting the honour and expressed a desire to know more about Hamilton's new results. The latter then wrote the following letter to Lloyd on the 10th (MS No. 299):

Just after the evening when I gave to the R. I. Academy an account of my late optical results I wrote to Professor Airy and among other things I mentioned that I had arrived at a new consequence from Fresnel's theory without stating what that consequence was. I now enclose a letter received from him yesterday in which he expresses a wish to be informed of it: and if you should, as you seemed to think likely, be prevented by want of apparatus or leisure from making soon any decisive experiment on the point, I believe it will be well to mention the theoretical result to Airy.

To this Lloyd replied (MS No. 300):

I fear it would be wholly impossible to obtain experimentally any decisive result connected with your theoretical conclusion, without better means than I have at present at my disposal. The angle of divergence produced by diffraction in the minutest apertures when they are so close as they must be in any specimen, is far greater than the angle we seek. The specimens I showed you the other day are fine but I find they belong to a form of crystallisation which the mineralogists call macled [*sic*], that is in fact they are composed of several distinct crystals crossing each other. They would therefore be wholly unfit for the purpose. I am sure your conclusion can be readily tested by anyone having access to fair specimens, but as

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> this is not the case here, you had better refer the matter to Airy or someone else as soon as possible.

Hamilton's impatience did not prevail however and he never did inform Airy of his discovery before the phenomenon had been observed by Lloyd on 14 December. On that day Lloyd wrote to Hamilton (MS No. 305):

> I write this line to say that I have found the *cone*, at least I have almost no doubt on the subject but will still verify it by different methods of observation. I have no time to say more at present than that I have observed it in a fine specimen of arragonite, which I received from Dollond in London since I saw you last.

The two men met on the morning of 18 December and at 3 o'clock on that Tuesday afternoon Lloyd wrote the following happy note to his colleague (MS 307):

> I am happy to tell you that since I saw you this morning I succeeded in projecting the cone on a screen of roughened glass; and, observing a section of it so large as 2 inches in diameter, you will easily conceive that the phenomenon is most striking; the appearance is exactly the same as that we saw when *looking through* the aperture.
>
> Its deviation from an exact circle, however, is of course more distinctly seen. I traced the boundary of the section on the screen and then measured the distance as accurately as I could. Three such measurements gave me for the angle of the cone 6°24′, 6°22′, 5°56′, which you see are tolerably near. The mean (6°14′) corresponds pretty well with the measurements of the extreme circle, taken yesterday. The difference between it and the theoretical result is probably the effect of diffraction, and I must now try and correct for this *perturbation*. This mode of exhibiting the phenomenon is decisive as well as beautiful, and I am sure you will be glad to see it when you next come to town.

The next day Hamilton replied as follows (MS No. 308):

> I am very glad to find by your note of yesterday that you are so vigorously and successfully pursuing your experiments. I on my part am at work reading and thinking on the dynamics of light and on other connected subjects. Since I saw you yesterday I wrote to Herschel and Airy and mentioned that in seeking to verify my theoretical conclusions respecting conical refraction you had discovered a new and curious class of optical phenomena. When you are disposed to draw up any account of your experiments, if you favour the RIA with it, I will apply to them for leave, and am sure the leave will be given, to publish a note in the Annals of Philosophy or some such place, without waiting for the slow appearance of our volume.

Airy replied to Hamilton's letter on 23 December (MS No. 309). Having accepted and expressed his gratitude for the honorary membership of the Royal Irish Academy, he turned to the discovery of conical refraction. He writes:

> I am very much interested with your discovery of the circular contact of the tangent plane with Fresnel's double wave surface.
>
> I was well aware (a long time ago) that the point of the surfaces, which in the principal section is the intersection of the circle and the ellipse, is in the surfaces the meeting of two *dimples* (external and internal), and that these dimples near their point of meeting become ultimately two opposite cones; the outer one diverging in a sort of trumpet mouth. But I had no idea that the mouth of the trumpet could be touched by one plane. Now as to the consequences of this I am extremely puzzled . . . Arragonite is a bad substance, I should imagine; I should think topaz likely to make a wider cone, perhaps your formulae will show you at once. Let me beg you to communicate as soon as possible (if Professor Lloyd does not object) the phenomena which he has observed.

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This letter shows that, however close Airy may have been to the discovery of the phenomenon, he had missed the essential points about the tangent planes and tangent cones to the wave surface. He also had not contemplated the cusp ray which was the key to Hamilton's discovery. Furthermore, the above letter makes clear that at this point Hamilton had informed Airy only in very general terms about the discovery. Airy's remark about arragonite being a bad substance is not correct. Hamilton resumed correspondence with Airy early in the new year. On 4 January he communicated at length the results of Lloyd in the investigation of external conical refraction, together with some remarks of his own with respect to the vibrations, interference and polarisation involved in the experiments. In his reply of the 16th Airy expressed strongly the view that, if the phenomenon of external conical refraction be true, it has no connection with the theory outlined by Hamilton. The latter wrote again on 21 January and on 1 February to clarify the results which had been obtained. Then, having dispatched the second letter, he received a letter from Airy dated 28 January, which shows that the penny had finally dropped. Airy writes (MS No. 333):

> Allow me to thank you for your last note, which is all comprehensible and true; and if I had not been very dull, I might have guessed at some of it before. You had not mentioned to me anything about the cusp ray, and therefore there were parts of the previous letter which were altogether mysterious to me, and were likely to remain so, except I could divine or you explain.

A series of letters now passed between Hamilton at Dunsink Observatory and Lloyd in Trinity College during the first week of January which show clearly the course of development of Hamilton's ideas which later appeared in print in the completed 'Third supplement'. The first of these letters was written by Hamilton on 2 January (MS No. 312).

> A hasty investigation, in which however I have not purposely omitted any term as small on account of the small eccentricities of the ellipsoid, has led me to the conclusion that vibrations on Fresnel's wave infinitely near an optic axis, that is, infinitely near a cusp, are in the normal planes to the wave which contains the optic axis; and that such a normal plane revolves exactly half as fast about the optic axis as the plane containing the axis and the infinitely near radius vector or ray. These conclusions you understand to be obtained by passing to the limit of the distance from the cusp, but not by supposing any approach to equality between the constants *a, b, c*. I find also that the planes of polarisation of the infinitely near rays all intersect in one common line which is in the plane of the two optic axes and is the normal to the ellipse at the cusp; the plane of polarisation of a ray being defined to be perpendicular to the vibration; so that all the vibrations infinitely near the cusp are parallel to one plane which contains the tangent of the ellipse, and is perpendicular to the plane of the optic axes.
>
> Combining these results, which I believe to be rigorous, with the approximate equality of *abc*, we get approximately a law for the plane of polarisation analogous to that which we had both obtained for the other question.
>
> You may like to have my October equation of . . . the cone of tangents; it is with a proper choice of co-ordinates
>
> $$z = b - \frac{\sqrt{a^2 - b^2}\sqrt{b^2 - c^2}}{2\,abc}(x + \sqrt{x^2 \pm y^2});$$
>
> the origin is at the centre of the wave, one optic axis is the axis of z, and the other optic axis is in the plane of xz.

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This equation appears in the 'Third supplement' in an equivalent form with respect to the rectangular coordinates $x_{,}, y_{,}, z_{,}$, where the plane of $x_{,}\, z_{,}$ is the plane of ac and the positive semiaxis of $z_{,}$ coincides with the line ρ' of single ray velocity or the cusp ray, viz.

$$z_{,} = b - \frac{1}{2} b^2 \sqrt{c^{-2} - b^{-2}} \sqrt{b^{-2} - a^{-2}} \, (x_{,} \pm \sqrt{x_{,}^2 + y_{,}^2}).$$

The error in the equation of the letter has been removed. He continues:

> And for the emergent refracted cone of rays in vacuo, corresponding to perpendicular incidence on plane face $z = b$, I found rigorously
>
> $$\sigma^2 + \tau^2 = \frac{\sigma\sqrt{a^2 - b^2} \, . \sqrt{b^2 - c^2}}{abc};$$
>
> σ being the cosine of inclination of emergent ray to the axis of x, τ to the axis of y.
>
> The coefficient $\dfrac{\sqrt{a^2 - b^2} \, . \sqrt{b^2 - c^2}}{abc}$ is the tangent of the angle at which the ellipse and the circle intersect: it is also the sine of the extreme divergence in my emergent cone, for perpendicular incidence.

The next letter (MS No. 313), written on 3 January, shows that Humphrey Lloyd was well acquainted with the mathematical and theoretical aspects of the question; here he points out an error in the denominator of the equations in the previous letter. He writes:

> You mention in your note of yesterday that the tangent of the angle at which ellipse and circle intersect is
>
> $$\frac{\sqrt{a^2 - b^2} \, . \sqrt{b^2 - c^2}}{abc}$$
>
> the coefficient in your equations. I had occasion to look for that angle while enquiring into the law of the planes of polarisation, and find its tangent to be
>
> $$\frac{\sqrt{a^2 - b^2} \, . \sqrt{b^2 - c^2}}{ac}$$
>
> differing from yours by the factor b. As I find no defect in my demonstration, I am greatly puzzled to account for the difference.

At the bottom of this manuscript there is a calculation in Hamilton's hand, where he checked the coefficient in question and discovered that Lloyd was in fact correct, and on 3 January he wrote the following brief note (MS No. 315).

> You are certainly right and I am wrong about the angle between the circle and the ellipse. I must examine how I made the mistake. I have great hopes that I have made the same mistake in the angle of the cone, and if so we shall bring up the theoretical angle to nearly 5°.

This note was written in haste as Hamilton was packing for the Limerick mail, and he promised to write again on the same subject the next day. At this time Hamilton was

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courting Helen Bayly, daughter of the rector of Nenagh in county Tipperary. They were married on 9 April 1833. There are therefore a number of letters written from Nenagh in early 1833.

The first letter from Tipperary is dated 4 January (MS No. 316). He writes:

I hope you received a note which I wrote in great haste yesterday before I started from Dublin to thank you for pointing out my mistake about the angle of intersection of circle and ellipse. A very simple reasoning indeed ought to have shown one that the angle could only be a function of the ratios of a, b, c, and therefore that my formula must have been wrong. In like manner my equation of the cone of tangents (from which I deduced the angle) was obviously wrong, since it ought to have given z as a homogeneous function of the first dimension of a, b, c, x and y. In the mail last night I went over the whole process of transformation of co-ordinates in my thoughts, and having thus prepared the equation of the wave, I deduced the cone of tangents as follows.

$$z = b - \frac{\sqrt{a^2 - b^2} \cdot \sqrt{b^2 - c^2}}{2ac}(x \pm \sqrt{x^2 + y^2}),$$

differing from my former equation by not having b in the denominator.

But the same mental calculation confirmed my old results about the equation of the cone of rays, emerging from the perpendicular face $z = b$, and gave me again the equation

$$\sigma^2 + \tau^2 = \frac{\sqrt{a^2 - b^2} \cdot \sqrt{b^2 - c^2}}{abc}\sigma;$$

σ, τ having the same meanings that I mentioned in my recent note, namely $\sigma =$ cosine of inclination of emergent ray to positive semiaxis of x (which touches circle and is above ellipse) and $\tau =$ cosine of same emergent ray to positive semiaxis of y, which is perpendicular at the cusp to the plane of optic axes. So, if I rightly remember what I said in my note of the other morning, the necessary conditions will be these: b is to be erased in the denominator of coefficient in equation of cone of tangents; and the words *mean index* $(1/b)$ *multiplied by* to be inserted before *tangent of angle of inclination of circle and ellipse* in the sentence comparing that tangent with the sine of extreme divergence of rays in their own cone.

The results about polarisation remain (I think) unaltered. I have looked over (just now) the sheets of my Supplement which bear upon the points, but find that the mistake in the equation of the cone of tangents appears to be an isolated slip: the transformed equation of the wave from which I had deduced it seems accurate, and so do the formulae that come afterwards respecting conical refraction – in fact it was nearly *en passant* that I calculated the equation of the cone of tangents at all, and I never thought about the angle between the circle and ellipse till I was hastily writing to you the other morning: for my general methods of treating refractions did not at all require me to know that angle nor that equation, nor to think expressly of the existence of a conoidal cusp.

I use, to determine the refracted rays in air, the two following parallel formulae of my own for incidence on a face parallel to xy,

$$\sigma = \frac{\delta v}{\delta \alpha}, \quad \tau = \frac{\delta v}{\delta \beta},$$

in which σ and τ are the cosines already described; v is the reciprocal of the undulatory slowness of the incident ray in the crystal, expressed as a homogeneous function of the cosines α, β, γ of the angles between the ray and the rectangular axes

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of x, y and z. And in my Supplement I had arrived (I still think rightly) at the following approximate expression for ν in the present equation

$$\nu = b^{-1} + \frac{1}{2} b. \sqrt{b^{-2} - a^{-2}} . \sqrt{c^{-2} - b^{-2}} (\alpha \pm \sqrt{\alpha^2 + \beta^2})$$

which gave, by the meanings of my symbols $\frac{\delta \nu}{\delta \alpha}, \frac{\delta \nu}{\delta \beta}$, ($\alpha, \beta, \gamma$ being treated as if they were independent) and by my conditions of refraction

$$\sigma = \frac{b}{2} \sqrt{b^{-2} - a^{-2}}. \sqrt{c^{-2} - b^{-2}} . \left(1 \pm \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \right),$$

$$\tau = \pm \frac{b}{2} \sqrt{b^{-2} - a^{-2}}. \sqrt{c^{-2} - b^{-2}} . \frac{\beta}{\sqrt{\alpha^2 + \beta^2}},$$

expressions for an emergent ray which grow independently more accurate as α, β diminish, that is, as interior incident ray approach to optic axis: and hence by eliminating β/α I obtained the equation given above for the emergent cone.

These expressions (for σ, τ, υ and ν) were introduced in the printed 'Third supplement' with no further explanation to that given in the letter above. In the 'Third supplement' the quantities σ, τ, υ, are introduced as the 'components of normal slowness', being given as the partial differential coefficients of the characteristic function (V) with respect to x, y, z. A fundamental partial differential equation

$$\left(\frac{\delta V}{\delta x}, \frac{\delta V}{\delta y}, \frac{\delta V}{\delta z} \right) = 0$$

is then constructed, which the function V must satisfy in a uniform medium. The surface of components is arrived at from consideration of this equation. The quantities σ, τ, υ are also defined in terms of the 'slowness of the ray'. Thus they are written as the partial differential coefficients of ν with respect to the cosines α, β, γ, of the angles between the ray and the rectangular axes x, y, z. The long letter of 4 January continues:

> Altho' I cannot find therefore that my error of the b, in the equation of the cone of tangents, has extended beyond that equation, and tho' I trust I should have detected it before it went to press (as I always verify my results in various ways before I print them) yet I am much indebted to you for so early showing me my mistake. If I had made the same mistake in the emergent cone, the effect would not be of the kind that I thought in my hasty note of yesterday.
>
> As you have logarithms at hand it might be worthwhile to verify the numeric results of my formula which however properly belongs only to perpendicular incidence. I mean

$$\sin = \frac{\sqrt{a^2 - b^2} . \sqrt{b^2 - c^2}}{abc}.$$

Lloyd's reply of 5 January (MS No. 317) reveals the progress of his experiments, reported a month or so later in the *Philosophical Magazine*. This letter contains a diagram showing a dark circle or spot surrounded by a bright ring.

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I received your letter this morning and have to thank you for the valuable information it contains. As to the angle between circle and ellipse and the equation of the cone of tangents, the slip was too obvious to have escaped your eye. I have calculated

$$\sin^{-1}\frac{\sqrt{a^2-b^2}\,.\,\sqrt{b^2-c^2}}{abc},$$

the angle of the cone for perpendicular incidence and found it to be 2°57′. But I have no dependence on my arithmetical accuracy. In the actual case which I have submitted to observation the incidence within the crystal is 1/2 the angle of the optic axes or 10° nearly and the resulting cone must be *a little larger*. I am glad that you have not been able to make any change in the amount of this angle on either side, for I find on reviewing my measurements, that they approach it *much nearer* than I had expected. From some simple considerations I conclude that the *true* angle of the cone is 1/2 *the sum* of the angles of the inner and outer observed conical surface. I have consequently rejected all measurements in which I have taken no account of the angular magnitude of the *interior dark space*. I have remaining five measurements which thus corrected range from 3° to 4° and their mean is 3 1/2°, which is as near as could be expected. I place most reliance on the measurements which I took when the cone was projected on a screen but from the state of the weather have unfortunately not been able to repeat them.

I believe I told you of some interesting variations in the phenomena when apertures of different sizes were employed. I find that they are all explicable on the simplest principles. In my proof of the law of the planes of polarisation I have assumed that a ray indefinitely near the optic axis *within* the crystal will be divided into two at emergence, the plane of which coincides q.p. with the plane passing through the interior ray and the optic axis. I believe this is approximately the law, at least when the surface of emergence is perpendicular to the axis. Have you not made a similar supposition? Did your 3° belong to the case of perpendicular incidence or to the incidence of 10°? I do not think there will be much difference between the results in the two cases. I am sure I fully comprehend your wishes as to the mode in which you desire your discoveries to be noticed in the paper which I am about to send to the *Philosophical Magazine*, so if you are not in town before I send it off, I am sure you will be satisfied when you see it in print.

In his reply of 6 January, Hamilton referred to the first letter of the series, written on 3 November, to answer the queries in Lloyd's letter. He writes (MS No. 318):

I have just received your letter and luckily find among my papers a memorandum of some calculations which I made two months ago, with respect to the emergent cone from arragonite when the face is (as I take it to be) the plane of *bc*: I find also a memorandum of a note to you on the subject, dated 3 Nov, 1832, which if you can lay your hands on it, will answer most questions about oblique incidence.

But to save you the trouble of a search, I may now copy my formulae for the two angles in air, measured from the normal to the face, which normal (as well as the two angles in question) is supposed in the plane of ac: they are incident ray being cusp ray,

$$\text{Sin } R_o = 1{,}6863, \text{ Sin I}$$

$$\text{Sin } R_e = 1{,}68708, \text{ Sin } (I - 1°44'48'');$$

I is here the internal angle of incidence, and R_o and R_e, are the corresponding angles of refraction in air, of which the difference may be called the angle of the cone. When $I = 0$, I found this angle $= 2°56'51'' = R_o - R_e$, agreeing with your 2°57′;

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and [when] $I = 9°56'27''$ (that is a normal to face) then the angle of cone = $16°55'27'' - 13°54'49'' = 3°0'38''$. It is here supposed that 1/a = 1,5326; 1/b 1,6863; 1/c = 1,6908; according to the numbers that you gave me. The 1°44'48" which I gave in November in the formula for R_e, is, by the nature of the question, the angle between the circle and the ellipse: and as it was calculated by the correct formula tan =

$$\frac{\sqrt{a^2 - b^2}.\sqrt{b^2 - c^2}}{ac},$$

it shows that my calculations respecting the *rays* were ever consistent with the correct formula. As to the coefficient 1,68708 in the formula for R_e, it is equal to the reciprocal of the perpendicular let fall from the centre of the wave, on the tangent drawn to the ellipse at the cusp. But I deduced this coefficient, and the angle 1°44'48", in November by my general algebraical methods, without expressly thinking of this tangent to the ellipse. For the case of perpendicular internal incidence in the direction of a cusp ray, I find that an internal infinitely near ray is divided into two on emergence, but that the two planes containing these and the cusp ray do not coincide with each other nor with the plane containing incident ray and cusp ray.
On the contrary, the two planes first mentioned are (I think) perpendicular to each other, and bisect the acute and obtuse angles formed by the other plane (of incidence) and the plane of optic axes: because in my formula

$$\sigma = \frac{\sqrt{a^2 - b^2}\,.\,\sqrt{b^2 - c^2}}{2abc}\left(1 \pm \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}\right)$$

$$\tau = \frac{\pm\sqrt{a^2 - b^2}\,.\,\sqrt{b^2 - c^2}}{2abc}\cdot\frac{\beta}{\sqrt{\alpha^2 + \beta^2}},$$

β/α is the tangent of the azimuth of the plane of incidence, and τ/σ is the tangent of the azimuth of the plane of refraction, counted from plane of optic axes and these tangents are connected by the relation

$$\frac{\tau}{\sigma} = \frac{\pm\beta}{\sqrt{\alpha^2 + \beta^2} \pm \alpha},$$

so that instead of being equal, the azimuth of refraction is either half the azimuth of incidence, or else 90° + half that incident azimuth.

Is this result opposed to any of yours?

If so, and if you write again . . . I shall be able to send you another letter after considering the subject. I mentioned that my results respecting polarisation are unconnected with the slip about the angle between circle and ellipse, and are I believe correct.

More than two weeks elapsed before the next letter, by which time Hamilton had returned to Dublin. On 23 January he wrote to Lloyd on the subject of internal conical refraction (MS No. 328):

I have no logarithms by me, but being desirous of comparing your late experiments on internal conical refraction, with my theoretical expectations, I have reasoned thus since I saw you.

My theoretical angle of the internal cone, that is, extreme internal angular deviation from that internal ray which is normal to a circular section, has

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$$\text{tang} = \frac{\sqrt{a^2 - b^2} \cdot \sqrt{b^2 - c^2}}{b^2};$$

and a certain other angle (which I have since perceived to be the angle of intersection of circle and ellipse, for arragonite) was given by me in our early communications as 1°44′48″ being deduced from the formula

$$\text{tang} = \frac{\sqrt{a^2 - b^2} \cdot \sqrt{b^2 - c^2}}{ac}.$$

Because angle of internal cone = nearly $ac/2 \times 1°44'48''$; and with the same indices ($a^{-1} = 1{,}5326$; $b^{-1} = 1{,}6863$; $c^{-1} = 1{,}6908$) we have very nearly $ac/b^2 = 1{,}1$; and because finally, the theoretical angle of the internal cone = 1°55′ and your experimental angle was I think 1°52′, agreeing as clearly as can be desired.

It will be curious to have your experimental establishment of the two kinds of conical refraction coming before the Academy at the evening meeting next following that at which I gave my theoretical announcements.

On 28 January 1832 Humphrey Lloyd read at the Academy the paper which was subsequently published in the *Transactions* [14]. Lloyd's account of the discoveries was also published in the *Philosophical Magazine*, in Poggendorff's *Annalen*, in the *Annales de Chimie* and in the *British Association Report* for 1833. As the discovery of conical refraction was a joint discovery it can best be dated by the presentation of the papers at the Academy on 22 October 1832 and 28 January 1833 by Hamilton and Lloyd respectively.

The completed scenario of the discovery is revealed in two letters which Hamilton wrote to Herschel on 18 December (MS No. 306) and on 29 January (MS No. 335). Finally these two letters are presented. In the first Hamilton writes:

You are aware that the fundamental principle of my optical methods does not essentially require the adoption of either of the two great theories of light in preference to the other. However, I naturally feel an interest in applying my general methods to Fresnel's theory of biaxial crystals; and when in October I was finishing my *Third Supplement* for the Royal Irish Academy, I deduced from such application, some results respecting the focal lengths and aberrations of lenses formed of such crystals. In the course of these calculations I was led to transform in various ways Fresnel's law of velocity, or in other words, to study his curved wave: and I found, what he seems to have not suspected, that the wave has 1st, *four cusps* (at the end of the optic axes) at each of which the tangent planes are (not, as he thought, two, but) infinite in number; and 2nd, four circles of plane contact, along each of which the wave is touched, in the whole extent of the circle, by a plane (parallel to one of the circular sections of the surface of elasticity); somewhat as a plum can be laid down on a table so as to touch and rest on the table in a whole circle of contact, and has, in the interior of the circular space, a sort of conical cusp. Hence I was led to expect that under certain circumstances, easily deduced and assigned by me from these geometrical properties, a single incident and unpolarised ray would undergo not double but *conical refraction*.

I announced this expectation to the Royal Irish Academy at their monthly meeting in October, when I was giving an account of the results of my Third Supplement; and I applied to Professor Lloyd, son of our Provost here, to submit the matter to experiment. For some time he could do nothing decisive, not having any biaxial crystals of sufficient size and purity; but having lately obtained from Dollond a fine piece of arragonite, and having treated it according to my theoretical

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indications, he has conceived a curious and beautiful set of new phenomena, which, so far as they have been yet examined, appear to agree with theory, and at any rate are worthy of study. I thought this intelligence would interest you.

In the second letter six weeks later he writes:

Professor Lloyd read to the Royal Irish Academy last night a paper, 'On the phenomena presented by light in its passage along the axes of biaxial crystals', in which he gave an account of some recent additional experiments confirming my theoretical conclusions respecting conical refraction. These conclusions are chiefly the following

1st. A single plane wave within a biaxial crystal parallel to a circular section of the surface of elasticity, corresponds in general to an infinite number of internal ray directions; in such a manner that a single incident ray in air will give an internal cone of rays (of the second degree) and will emerge (from a plane face) as an external cylinder of rays if the external incident wave have that direction which corresponds to the foregoing internal wave. In this kind of *internal conical refraction,* one refracted ray of the cone is determined by the ordinary law of sines, using the mean index $1/b$; and the greatest angular deviation in the cone from this ray is in the plane of the optic axes and is

$$= \tan^{-1} \frac{\sqrt{a^2 - b^2} \, . \sqrt{b^2 - c^2}}{b^2} = 1°55' \text{ for ray } E$$

in arragonite, if we use Rudberg's elements.

Professor Lloyd has lately observed an emergent cylinder corresponding to this theory; from his measurements upon which the angle of the cone appeared to be 1°52′. He used a fine piece of arragonite procured from Dollond, thickness = 0,49 inch; the incident ray was of solar light and it was passed through two small holes, the first in a screen at some distance from the crystal, the second in a thin metallic plate adjoining the first surface of the crystal; the emergent cylinder of rays was received on silver paper and produced on the paper a small white annulus, of which the size was the same at different distances of the paper from the arragonite. The emergent light was polarised according to a law which agrees with Fresnel's principles. Great care was necessary in the adjustment of the holes; when the adjustment was slightly disturbed, two opposite quadrants of the circle appeared more faint than two others and the two pairs were of completely complementary colours.

2nd. I concluded also from Fresnel's principles that a single internal cusp ray (often called an *optic axis* but not normal to a circular section of the surface of elasticity and on the contrary normal to a circular section of Fresnel's ellipsoid, one of those two rays of which each has but a single value for the velocity of light along it) ought, on emerging into air, to undergo, not *bifurcation* as Fresnel thought but (external) conical refraction. If the internal incidence be perpendicular, the equation in rectangular co-ordinates of the emergent cone may be put under the form

$$\frac{x^2 + y^2}{x \, . \sqrt{x^2 + y^2 + z^2}} = \frac{\sqrt{a^2 - b^2} . \sqrt{b^2 - c^2}}{abc} = \text{Sin } 2°57'$$

for ray E, with Rudberg's elements for arragonite; this cone, therefore, is of the fourth degree (whereas the internal was of the second) but it does not differ much from a circular cone. In Professor Lloyd's experiments the normal to the refracting face was Fresnel's axis *a* bisecting the acute angle between the two cusp rays, and the internal incidence was therefore about 10°; which made the theoretical angle of the emergent cone somewhat more than 3°, instead of 2°57′. He has sent to the

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> Annals of Philosophy a sketch of his experimental results, which appear to agree sufficiently with the theory, as to the position and magnitude and polarisation of the emergent cone in the external conical refraction. More lately he has taken new measures which appear to agree still better and he had made those experimental verifications which I have attempted in this letter to describe, of the other (the internal) kind of conical refraction. The appearances in direct vision, or when the light is received on a screen, are interesting enough and vary pretty well with the shape and size of the aperture in the phenomena of external conical refraction. Figures will be given in the fuller Memoire in the Transactions of our Irish Academy. The experimental establishment of these new consequences from Fresnel's principles, must I think, be considered as interesting.

By 'Annals of Philosophy' Hamilton intended of course the *Philosophical Magazine*. The indices of refraction for arragonite used by Lloyd and Hamilton and referred to above are those determined by Frederik Rudberg and published in 1831 and 1832 [20].

Conclusion

From the correspondence presented above it is clear that the discovery of conical refraction, following the prediction by Hamilton, was the result of close collaboration between the mathematician and the experimentalist Lloyd. It is also clear that Lloyd had a profound understanding of the mathematical results derived by Hamilton, that he even discovered mistakes in the formulae of the latter and assisted the latter in putting the final touches to his theory of conical refraction and polarisation. However, Hamilton also understood the practical implications of his discovery and was able to propose experimental arrangements for discovering the phenomena.

Sarton and Hankins [21], [10] have pointed out that at least two others, namely Airy and MacCullagh, came close to making the discovery. Here it is necessary to differentiate clearly between the theoretical prediction and the experimental discovery. It is evident from the correspondence between Hamilton and Airy that the latter had not understood the essential properties of the cusp rays and circles of contact and was not likely to discover the phenomena. Airy, like Fresnel, understood that there were four singularities on the wave surface but had not grasped their essential character. Furthermore, Airy was not engaged in the kind of mathematical investigation of the wave surface from which he might have discovered the new phenomena and developed a theory to account for them. It seems unlikely that the phenomena could have been discovered in the course of a purely experimental investigation of the arragonite crystal. On the other hand it is a reasonable conjecture that Airy might have made the discovery before Lloyd, had Hamilton informed him of the details of his discovery in October or November 1832.

The law of conical polarisation described by Lloyd in his published account is of interest and significance because unlike the phenomena themselves it was not predicted in advance by Hamilton, although he subsequently introduced it in the 'Third supplement' as if it had been predicted in the same way as the cones. Hankins has suggested that Airy had an inkling of some peculiar law of conical polarisation before Lloyd formulated and published the law in question [10, pp 92-3]. He believes that 'Airy had nearly walked off with at least a portion of the prize'. On the other hand the correspondence with Hamilton in January 1833 would seem to support the view that Airy was rather sceptical about conical refraction and its connection with Hamilton's

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theory. It was only on 28 January that he understood the phenomena and in particular the all-important role of the cusp ray. By this time Lloyd's account was already in the press. It seems unlikely that he might have formulated the law in advance of Lloyd.

As regards the theoretical prediction of conical refraction there are in fact good grounds for supposing that MacCullagh might have walked off with the whole prize. In his paper 'On the double refraction of light according to the principles of Fresnel', read at the Royal Irish Academy on 21 June 1830, and published the following year in the *Transactions,* he developed a series of geômetrical propositions which he then applied with great success to explain the nature and properties of the wave surface. Hankins has pointed out that Hamilton not only knew of the paper of MacCullagh but had written a review of this and another memoir of MacCullagh in the *National Magazine,* Dublin, in August of that year.

After the discovery of conical refraction MacCullagh claimed, in a note published in the *Philosophical Magazine* in July 1833, that 'it is an obvious and immediate consequence of the theorems published by me, three years ago'. MacCullagh's claim almost provoked an unpleasant priority dispute which was only obviated by the timely intervention and mediation of Humphrey Lloyd. The painful fact for MacCullagh was that he had not deduced the physical consequences from his geometrical investigation of the wave surface as he might so easily have done. His claim that he had contemplated conical refraction in advance of Hamilton and had intended publishing on the subject was hopeless.

Conical refraction is little more than a curious optical phenomenon which had no conceivable application. The significance of its discovery was entirely theoretical. It was a vindication of the Fresnel theory of double refraction and therefore was the last chapter in the history of the theory of double refraction, which had begun a century and a half before with Huygens. It was a further triumph for the wave theory of light over the particle theory and caused a good deal of excitement among the scientific community at home and abroad. In the history of physics it is one of the rare examples of a mathematical prediction being subsequently verified by experiment. For Hamilton it was the crowning achievement of his view and method of optics developed in the essay and supplements on the 'Theory of systems of rays'. Lastly, for the historian of science the discovery has provided a wealth of interesting and valuable material which continues to arouse the interest of scholars in the field.

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